INDEX OF SEQUENCES

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Abstract. Let Z_n denote the cyclic group of n elements. For every $x \in Z_n$, we denote by $|x|_n$ the smallest positive integer in the residual class x. Let $S = (a_1, \ldots, a_k)$ be a sequence of elements in Z_n. We say that S has index n if there is an integer m co-prime to n such that $\sum_{i=1}^{k} |ma_i|_n = n$. In this paper, we prove that a sequence of n elements in Z_n with repetition at least $\frac{n}{2}$ contains a subsequence with index n, and if $n = p$ is a prime the restriction of repetition can be relaxed to $\frac{p-2}{10}$. On the other hand, for every $n = 4k + 2 \geq 22$, we give a sequence of length $n + \lfloor \frac{n}{4} \rfloor - 5$ and with repetition $\frac{n}{2} - 1$ which contains no subsequence with index n. This disproves a conjecture given twenty years ago by Lemke and Kleitman.

Key Words: Index of sequences; Zero-sum; Minimal zero-sum sequence

1. Introduction and Notations

Let Z_n be the cyclic group of n elements. We can regard Z_n as the residual classes group of Z modulo n. For every $x \in Z_n$, let $|x|_n$ denote the smallest positive integer in the residual class x. Let $S = (a_1, \ldots, a_k)$ be a sequence of elements in Z_n. Let $\sigma(S) = \sum_{i=1}^{k} a_i$. We denote by $|S|_n$ the sequence $(|a_1|_n, \ldots, |a_k|_n)$ of positive integers. For a subset A of Z_n, we denote $S \cap A = \prod_{a_i \in A} a_i$ to be the restriction of S to A. By $S \subseteq A$, we mean that $S \cap A = S$. For any integer m, let $mS = (ma_1, \ldots, ma_k)$. Define

$$Index(S) = \min_{(m,n)=1} \{\sigma(|mS|_n)\}.$$

Let

$$\sum_{Index}(S) = \{\sigma(|T|_n) : T|S, 1 \leq \sigma(|T|_n) \leq n\}.$$

Let $h = h(S)$ be the maximal repetition value of S.

The concept of *Index* was introduced by Chapman, Freeze and Smith [2] in 1999, and research on *Index* can go back to Lemke and Kleitman [4] in 1989 when they made the following conjecture:

Conjecture 1.1. [4] *Let d, n be two positive integers with $d|n$, and let S be a sequence of n elements in Z_n. Then, there is a nonempty subsequence T of S and an integer m co-prime to n such that*

$$d\big|\sigma(|mT|_n)\big|n.$$

When $d = n$ Conjecture 1.1 becomes the following:

Conjecture 1.2. *If S is a sequence of n elements (repetition allowed) in Z_n then S contains a subsequence T with $Index(T) = n$.*

In this paper we demonstrate that Conjecture 1.2 and therefore Conjecture 1.1 is not true in general (Section 2), we prove that the conclusion of Conjecture 1.2 is true if S contains some element at least $n/2$ times (Section 3), and we also prove that if $n = p$ is a prime then the Conjecture is true if S contains some element at least $\frac{p-2}{10}$ times (Section 4). In fact, we still believe that Conjecture 1.2 is true when n is prime.

We call S a *zero-sum sequence* if $\sigma(S) = 0$, we call S a *minimal zero-sum sequence* if S is zero-sum and any proper subsequence of S is not zero-sum, and we call S a *zero-sum free sequence* if S contains no nonempty zero-sum subsequence.

Recently, the study of *Index* has attracted many researchers. Let S be a minimal zero-sum sequence in Z_n. It has been proven that if $|S| \le 3$ then $Index(S) = n$ (see [2]) and if $|S| \ge \frac{n}{2}+1$ then $Index(S) = n$ ([7], [8]). It has also been proven that for $5 \le k \le \frac{n}{2}$ there is a minimal zero-sum subsequence S in Z_n such that $|S| = k$ and $Index(S) \ge 2n$, and for $k = 4$ and $(n, 6) \ne 1$, there is a minimal zero-sum subsequence S in Z_n such that $|S| = 4$ and $Index(S) \ge 2n$. The following was conjectured.

Conjecture 1.3. [6] *Let n be a positive integer with $(n, 6) = 1$, and let S be a minimal zero-sum sequence in Z_n with $|S| = 4$. Then, $Index(S) = n$.*

Li and Plyley [5] proved that the above conjecture is true for $n = p$ a prime. We shall provide a new proof of this result in Section 5.

Our main results are the following:

Theorem 1.4. *Let S be a sequence of n elements in Z_n. If $h(S) < 4$ or $h(S) \ge n/2$ then S contains a subsequence T with $Index(T) = n$ and $|T| \le h(S)$.*

Theorem 1.5. *Let $p > 24318$ be a prime, and let S be a sequence of p elements in $Z_p \setminus \{0\}$. If $h(S) \ge \frac{p-2}{10}$ then S contains a subsequence with index p.*

Theorem 1.6. *Let $k \in \mathbb{N}$, $n = 4k + 2$, $G = \mathbb{Z}_n$. If $k \ge 5$, then*

$$S = 1^{\frac{n}{2}-3} \left(\frac{n}{2}\right) \left(\frac{n}{2}+1\right)^{\frac{n}{2}-1} \left(\frac{n}{2}+2\right)^{\lfloor\frac{n}{4}\rfloor-2}$$

contains no subsequence with index n.

2. Proof of Theorem 1.6

Proof of Theorem 1.6. Suppose the conclusion is false, i.e., there exists a zero sum subsequence T of S and $j \in [1, n-1]$ with $(j, n) = 1$ such that

$$\sigma(|jT|_n) = n. \tag{1}$$

Let

$$T = 1^x \left(\frac{n}{2}\right)^y \left(\frac{n}{2}+1\right)^z \left(\frac{n}{2}+2\right)^w,$$

where $x \le \frac{n}{2} - 3$, $y \le 1$, $z \le \frac{n}{2} - 1$ and $w \le \lfloor\frac{n}{4}\rfloor - 2$.

Then

$$\sigma(T) = (x + z + 2w) + \frac{n}{2}(y + z + w) \equiv 0 \pmod n. \tag{2}$$

Case 1. $j < \frac{n}{4}$.

Then

$$|jT|_n = j^x \left(\frac{n}{2}\right)^y \left(\frac{n}{2}+j\right)^z \left(\frac{n}{2}+2j\right)^w.$$

By (1), we have $y+z+w \le 1$. It follows that $\sigma(|T|_n) \le x + (\frac{n}{2}+2) \le \frac{n}{2} - 3 + \frac{n}{2} + 2 < n$, a contradiction.

Case 2. $\frac{n}{4} < j < \frac{n}{2}$.

Then

$$|jT|_n = j^x \left(\frac{n}{2}\right)^y \left(\frac{n}{2}+j\right)^z \left(2j-\frac{n}{2}\right)^w.$$

By (1), we have $x \le 3$ and $z \le 1$. Then, $x+z+2w \le 3+1+2\times(\lfloor\frac{n}{4}\rfloor - 2) < \frac{n}{2}$. Clearly, $x+z+2w > 0$. From (2) we derive that $x+z+2w \equiv 0 \pmod{\frac{n}{2}}$, a contradiction.

Case 3. $\frac{n}{2} < j < \frac{3n}{4}$.

Then

$$|jT|_n = j^x \left(\frac{n}{2}\right)^y \left(j-\frac{n}{2}\right)^z \left(2j-\frac{n}{2}\right)^w.$$

By (1), we have $x+y+w \le 1$. We claim

$$x+y+w = 1. \tag{3}$$

Otherwise, $x=y=w=0$ and $\sigma(T) = z + \frac{n}{2}z \not\equiv 0 \pmod{\frac{n}{2}}$, a contradiction with $\sigma(T) \equiv 0 \pmod{n}$.

Note that $0 < x+z+2w < n$. By (2), we have that

$$x+z+2w = \frac{n}{2} \tag{4}$$

and

$$y+z+w \equiv 1 \pmod{2}. \tag{5}$$

By (3) and (4), we have $y+z+w \equiv z+w-y = \frac{n}{2} - 1 \equiv 0 \pmod{2}$, a contradiction to (5).

Case 4. $\frac{3n}{4} < j < n$.

Then

$$|jT|_n = j^x \left(\frac{n}{2}\right)^y \left(j-\frac{n}{2}\right)^z \left(2j-\frac{3n}{2}\right)^w.$$

By (1), we have $x \le 1$ and $z \le 3$. Then, $x+z+2w \le 1+3+2\times(\lfloor\frac{n}{4}\rfloor - 2) < \frac{n}{2}$. Clearly, $x+z+2w > 0$. From (2), we derive a contradiction. □

3. Proof of Theorem 1.4

Lemma 3.1. [1] *Let S be a sequence of n elements in Z_n. Then S contains a zero-sum subsequence T with length $|T| \in [1, h(S)]$.*

Lemma 3.2. [6] *Let S be a minimal zero-sum sequence of elements in Z_n with $|S| \in \{1,2,3\}$. Then $Index(S) = n$.*

Proof of Theorem 1.4. Let $h = h(S)$. The case for $h < 4$ follows from Lemma 3.1 and Lemma 3.2. Thus, we need only to consider the case for $h \geq n/2$. Assume to the contrary that S contains no subsequence with index n and with length not exceeding h.

Write S in the form $S = a^h b_1 \cdot \ldots \cdot b_{n-h}$. If $o(a) < n$ then $o(a) \leq n/2 \leq h$. It follows that $a^{o(a)}$ is a subsequence of S with index n. So, we may assume that $o(a) = n$ and that (up to isomorphism)

$$S = 1^h b_1 \cdot \ldots \cdot b_{n-h}.$$

We show that for every $t \in [1, n-h]$ and any subset $\{i_1, \ldots, i_t\} \subset [1, n-h]$ the following holds

$$|b_{i_1}|_n + \cdots + |b_{i_t}|_n \leq n - h + t - 1. \tag{6}$$

We proceed by induction on t. For $t = 1$, assume to the contrary that $|b_i|_n \geq n-h+1$. Then $1^{n-|b_i|_n} b_i$ is a subsequence of S with index n and with length $n - |b_i|_n + 1 \leq h$, a contradiction.

Now assume that (6) is true for $t = k \in [1, n-h-1]$ and we want to show that (6) is also true for $t = k+1$. Let $\tau = |b_{i_1}|_n + \cdots + |b_{i_{k+1}}|_n$. By the induction hypothesis, $\tau - |b_{i_j}|_n \leq n-h+k-1$ holds for every $j \in [1, k+1]$. Therefore,

$\tau = \frac{1}{k}(k\tau) = \frac{1}{k}(\sum_{j=1}^{k+1}(\tau - |b_{i_j}|_n) \leq \frac{(k+1)(n-h+k-1)}{k} \leq n$ (since $k \in [1, n-h-1]$). If $\tau \geq n-h+k+1$, then $1^{n-\tau} b_{i_1} \cdot \ldots \cdot b_{i_{k+1}}$ is a subsequence of S with index n and with length $n - \tau + k + 1 \leq h$, a contradiction. This proves (6). Taking $t = n-h$ in (6) we have that $|b_1|_n + \cdots + |b_t|_n \leq n-h+t-1 = 2t-1$. This forces that $b_i = 1$ for some $i \in [1, n-h]$, a contradiction to $h = h(S)$. This proves the theorem. $\square$

Note that the sequence given in Section 2 has maximal repetition $\frac{n}{2} - 1$. So, the restriction that $h(S) \geq \frac{n}{2}$ in Theorem 1.4 is necessary for $n \equiv 2 \pmod 4$.

4. Proof of Theorem 1.5

Throughout this section, let $p > 24138$ be a prime and let S be a sequence of p elements in $Z_p \setminus \{0\}$. Up to isomorphism, we may assume that S contains 1 (note that 1 does not necessarily have the maximal repetition value).

Let $M = M(S)$ be the maximal integer t such that $\sum_{Index}(T) = [1, t]$ holds for some subsequence $T|S$. Let

$$m(S) = \max_{(r,p)=1} \{M(|rS|_p)\}$$

where r runs over all positive integers in $[1, p-1]$. For convenience, in the rest of this section, we shall always assume that

$$m(S) = M(S).$$

Lemma 4.1. *Let p be a prime, and let S be a sequence of p elements in $Z_p \setminus \{0\}$ with $1 \in S$. Put $M = M(S)$. Let T be a subsequence of S such that $\sum_{Index}(T) = [1, M]$. Then, $|T| \leq M$ and every term x of ST^{-1} satisfies $|x|_p \geq M + 2$. Furthermore, if $M = p$ or there is one term y of ST^{-1} with $|y|_p \geq p - M$ then S contains a subsequence with index p.*

Proof. $|T| \le |T|_p = M$. If there is some x of ST^{-1} satisfies $|x|_p \le M+1$, clearly, $\sum_{Index}(xT) = [1, M + |x|_p]$, a contradiction on the maximality of M. The second part of this lemma is clear. $\square$

Let $k \ge 2$ be a positive integer, and let $F[\frac{1}{k}, \frac{k-1}{k}]$ be all irreducible fractions between $\frac{1}{k}$ and $\frac{k-1}{k}$ and with denominators in $[2, k]$, i.e.

$$F\left[\frac{1}{k}, \frac{k-1}{k}\right] = \left\{\frac{a}{b} : (a, b) = 1, \frac{1}{k} \le \frac{a}{b} \le \frac{k-1}{k}, 2 \le b \le k\right\}.$$

Lemma 4.2. *Let $\frac{a}{b}$ and $\frac{c}{d}$ be two adjacent fractions in $F[\frac{1}{k}, \frac{k-1}{k}]$. If $\frac{a}{b} < \frac{c}{d}$ then, (i) $b+d \ge k+1$ and (ii) $bc - ad = 1$.*

Proof. (i) Note that $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$. Since $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent it follows that the irreducible fraction with value $\frac{a+c}{b+d}$ is not in $F[\frac{1}{k}, \frac{k-1}{k}]$. This forces that $b + d \ge k + 1$.

(ii) Since $(a, b) = 1$, there are two integers u and v such that $bu + av = 1$. Note that $b(u + ma) + a(v - mb) = 1$ holds for any integer m. Let $x = u + ma$ and $y = mb - v$. Then, $bx - ay = 1$. By choosing m suitably we may assume that $y \le k$ and $y + b \ge k + 1$. It follows that $y \ge k + 1 - b > 0$ and $x > 0$. From $bx - ay = 1$ we get

$$\frac{x}{y} - \frac{a}{b} = \frac{1}{by}.$$

If $y > 1$ then $\frac{x}{y}$ is a fraction in $F[\frac{1}{k}, \frac{k-1}{k}]$. So, either $\frac{c}{d} = \frac{x}{y}$ and we are done, or $\frac{c}{d} < \frac{x}{y}$. For the latter case we have

$\frac{1}{by} = \frac{x}{y} - \frac{a}{b} = (\frac{x}{y} - \frac{c}{d}) + (\frac{c}{d} - \frac{a}{b}) = \frac{b(dx-cy)+y(cb-ad)}{byd} \ge \frac{b+y}{byd}$. This implies that $d \ge b+y \ge k+1$, a contradiction.

Now assume that $y = 1$ and we must have $b = k$. It follows from $bx - ay = 1$ that $a = kx - 1$. Therefore, $x = 1$ and $a = k-1$. So, $\frac{a}{b} = \frac{k-1}{k}$ is the biggest fraction in $F[\frac{1}{k}, \frac{k-1}{k}]$, a contradiction. This completes the proof. $\square$

Lemma 4.3. *Let S be a sequence p elements in $Z_p \setminus \{0\}$ containing no subsequence with index p, and let $k = \lfloor \frac{p}{M} \rfloor$. Let us arrange all fractions in $F[\frac{1}{k}, \frac{k-1}{k}]$ increasingly. Let $\frac{a_i}{b_i}$ be the i-th fraction in $F[\frac{1}{k}, \frac{k-1}{k}]$ and let $f = |F[\frac{1}{k}, \frac{k-1}{k}]|$. Let $S_1 = S \cap [1, M]$, $S_2 = [M+2, \frac{p-1}{b_1}]$. For every $i \in [1, f]$, let $S_{2i+1} = S \cap [\frac{a_ip+1}{b_i}, \frac{a_ip+M}{b_i}]$ and let $S_{2i+2} = S \cap [\frac{a_ip+M+1}{b_i}, \frac{a_{i+1}p-1}{b_{i+1}}]$. Then, $S = \prod_{j=1}^{2f+1} S_j$.*

Lemma 4.4. *Let S be a sequence p elements in $Z_p \setminus \{0\}$ containing no subsequence with index p, and let f, S_i be defined as in Lemma 4.3. Suppose $4 \le M \le \frac{p-3}{2}$ and $\max\{\frac{p-M-2}{M}, \frac{p-M}{M+1}\} \le k \le \frac{p+1}{M}$. Then $|S_{2i+2}| \le b_{i+1} - 1$ for every $i \in [0, f-1]$. Furthermore, for every $i \in [2, k]$, define $R_i = \{x | S, 1 \le |ix|_p \le M, (|ix|_p, i) = 1\}$. Then we have,*

$$p = |S| \le M + \sum_{i=2}^{k} \sum_{(j,i)=1, 1 \le j \le i-1} (i-1) + \sum_{i=2}^{k} |R_i|.$$

Proof. If $i = 0$ then $S_2 = S \cap [M+2, \frac{p-1}{b_1}]$ and $b_1 = k$. If $|S_2| \ge b_1 = k$ then we can take a k-term subsequence U of S_2. Note that $p - 1 \ge |U|_p \ge k(M+2) \ge p - M$ and one can find a subsequence V of S_1 such that UV has index p, a contradiction. Now assume that $i \ge 1$. Now $S_{2i+2} = S \cap [\frac{a_ip+M+1}{b_i}, \frac{a_{i+1}p-1}{b_{i+1}}]$. If $|S_{2i+2}| \ge b_{i+1}$, take an arbitrary b_{i+1}-term

subsequence X of S_{2i+2}. Consider $|b_iS|_p$. It follows from Lemma 4.2 that $a_{i+1}b_i - a_ib_{i+1} = 1$, and so $b_i(\frac{a_{i+1}p-1}{b_{i+1}}) - a_ip = \frac{p-b_i}{b_{i+1}}$. Therefore, every term x of $|b_iS_{2i+2}|_p$ is in $[M+1, \frac{p-b_i}{b_{i+1}}]$ and $x \equiv -a_ip \pmod{b_i}$. Therefore, $p - b_i \geq |b_iX|_p \geq b_{i+1}(M+1) \geq p - b_iM$ (since by Lemma 4.2, $b_i + b_{i+1} \geq k+1$) and $|b_iX|_p \equiv -b_{i+1}a_ip = (1 - a_{i+1}b_i)p \equiv p \pmod{b_i}$. Therefore, there is a subsequence Y of S_1 such that $|b_i(XY)|_p = p$, a contradiction. This proves the first part of this lemma. For every $\ell \in [2,k]$, clearly, $R_\ell = \prod_{b_i=\ell} S_{2i+1}$. Therefore, $S = S_1 \prod_{i=0}^{f-1} S_{2i+2} \prod_{\ell=2}^{k} R_\ell$ is a disjoint decomposition of S into subsequences. Now the second part follows from the first part. $\square$

Lemma 4.5. *Let $n \in \mathbb{N}_{\geq 2}$, and let $a_1, a_2, \ldots, a_n$ (not necessarily distinct) be integers coprime to n. For any integer m, there exists a subset $\emptyset \neq I \subseteq [1,n]$ such that*

$$\sum_{i \in I} a_i \equiv m \pmod{n}.$$

Moreover, if $m \not\equiv 0 \pmod{n}$, we can choose such I with $\emptyset \neq I \subseteq [1, n-1]$.

Proof. We consider $a_1, \ldots, a_n$ as the elements in group Z_n. It suffices to prove that $\{a_1, 0\} + \cdots + \{a_{n-1}, 0\} = Z_n$. Let $H = St(\{a_1,0\} + \cdots + \{a_{n-1},0\})$. If $|H| = n$, then $|\{a_1,0\} + \cdots + \{a_{n-1},0\}| = n$ and the lemma follows. Now assume $|H| < n$. By Kenser's Theorem, we have that $|\{a_1,0\}+\cdots+\{a_{n-1},0\}| \geq \sum_{i=1}^{n-1} |\{a_i,0\}+H| - (n-2)|H| = (n-1)\times 2|H| - (n-2)|H| \geq n$, a contradiction. $\square$

Lemma 4.6. *Let p, S, M, k, R_i be defined as in Lemma 4.3, and let $2 \leq t < \ell < k$ with $d = \gcd(t,\ell) < t$. Let $u \in [2, M]$ be an integer. If $\frac{(t-d)p-\ell}{t\ell} \leq M \leq \frac{dp}{\ell} - t(u-1)$, then either $|R_t| = 0$ or $|R_\ell| \leq \frac{p-\ell M-2\ell+1}{u} + 2\ell - 1$.*

Proof. Suppose $|R_t| > 0$. Take an arbitrary term x of R_t. By the definition of R_t we get

$$\ell|x|_p \in \bigcup_{\gcd(i,t)=1, 0<i<t} [\frac{\ell ip+\ell}{t}, \frac{\ell ip + \ell M}{t}],$$

and thus,

$$\begin{aligned}|\ell x|_p &\in \bigcup_{d|i, 0<i<t} [\tfrac{ip+\ell}{t}, \tfrac{ip+\ell M}{t}] \\ &\subseteq [\tfrac{dp+\ell}{t}, \tfrac{(t-d)p+\ell M}{t}] \subseteq [p-\ell M, p-\ell(u-1)].\end{aligned}$$

If $||\ell R_\ell|_p \cap [1,u-1]| \geq \ell$, by Lemma 4.5 and the definition of R_t, we may choose a subsequence W of R_ℓ of length at most ℓ such that $|\ell W|_p \subseteq [1, u-1]$ and $|\ell x|_p + \sigma(|\ell W|_p) \equiv p \pmod{\ell}$. Since $\sigma(|\ell W|_p) \leq \ell(u-1)$, we have $|\ell x|_p + \sigma(|\ell W|_p) \in [p-\ell M, p]$. Thus, we can construct a subsequence of xWS_1 of index p, a contradiction. Therefore,

$$||\ell R_\ell|_p \cap [1, u-1]| \leq \ell - 1. \tag{7}$$

If $|R_\ell| < \ell$ then we are done. Otherwise, by Lemma 4.5, we get a subsequence R_0 of R_ℓ with $\sigma(|\ell R_0|_p) \equiv p \pmod{\ell}$ and

$$|R_0| \geq |R_\ell| - \ell. \tag{8}$$

We show that

$$\sigma(|\ell R_0|_p) \leq p - \ell M - \ell. \tag{9}$$

Otherwise, $\sigma(|\ell R_0|_p) \geq p - \ell M$, choose T to be the minimal subsequence of R_0 such that $\sigma(|\ell T|_p) \geq p - \ell M$ and $\sigma(|\ell T|_p) \equiv p \pmod{\ell}$. If $\sigma(|\ell T|_p) \leq p$, then we can construct a subsequence of TS_1 with index p, a contradiction. If $\sigma(|\ell T|_p) > p$, note that every term y of R_ℓ satisfies $1 \leq |\ell y|_p \leq M$ and $\gcd\{|\ell y|_p, \ell\} = 1$. By Lemma 4.5, by dropping at most ℓ terms from T, we get a proper subsequence $\tilde{T}$ such that $\sigma(|\ell \tilde{T}|_p) \geq p - \ell M$ and $\sigma(|\ell \tilde{T}|_p) \equiv p \pmod{\ell}$, a contradiction with the minimality of T. Therefore, $\sigma(|\ell R_0|_p) \leq p - \ell M - \ell$. By (7), we have that $\sigma(|\ell R_0|_p) \geq 1 \times (\ell - 1) + u \times (|R_0| - \ell + 1)$. By (9), then $|R_0| \leq \frac{p - \ell M - 2\ell + 1}{u} + \ell - 1$. Now the lemma follows from (8). $\square$

Lemma 4.7. *Let p, S, M, k, R_i be defined as Lemma 4.3. For any $t \in [2, k]$, let $1 = a_1 < a_2 < a_3 < \cdots$ be all positive integers coprime to t. If $M \leq \frac{p - 2t + wa_{u+1} + 2}{t + \sum_{i=2}^{u} a_i}$ for some $w, u \in \mathbb{N}_0$, then*

$|R_t| \leq \frac{p - (t + \sum_{i=2}^{u} a_i)M - 2t + 2}{a_{u+1}} + \delta_u(u-1)M + 2t + w$, where $\delta_u = 0$ for $u = 0$ and $\delta_u = 1$ for $u \geq 1$.

Proof. Suppose $|R_t| > \frac{p - (t + \sum_{i=2}^{u} a_i)M - 2t + 2}{a_{u+1}} + \delta_u(u-1)M + 2t + w$. It follows from $M \leq \frac{p - 2t + wa_{u+1} + 2}{t + \sum_{i=2}^{u} a_i}$ that $|R_t| \geq 2t + 1$. By Lemma 4.5, there exists a nonempty subsequence R_0 of R_t with

$$\sigma(|tR_0|_p) \equiv p \pmod{t}$$

and

$$|R_0| \geq |R_t| - t. \tag{10}$$

Similarly to Lemma 4.6, we can prove that

$$\sigma(|tR_0|_p) \leq p - tM - t. \tag{11}$$

Note that tR_0 contains $a_1 = 1$ at most $t - 2$ times, otherwise, $m(S) \geq M(tS) \geq tM + t - 1 > M$, a contradiction with $m(S) = M(S)$. Since $v_{a_i}(S) \leq h(S) \leq M$ for all $i \geq 2$, it follows that $\sigma(|tR_0|_p) \geq 1 \times (t-2) + \sum_{i=2}^{u} a_i \times M + a_{u+1} \times (|R_0| - (u-1)M - (t-2))$. By (11), we have $|R_0| \leq \frac{p - (t + \sum_{i=2}^{u})M - 2t + 2}{a_{u+1}} + \delta(u-1)M + t - 2$. By (10), we derive a contradiction. $\square$

Now we are in a position to prove Theorem 1.5.

Proof of Theorem 1.5. Assume to the contrary that S contains no subsequence with index p. Clearly, $M \leq p - 1$. By Lemma 4.1, every term x of ST^{-1} satisfies

$$M + 2 \leq |x|_p \leq p - M - 1.$$

This gives

$$M \leq \frac{p-3}{2}.$$

We distinguish several cases.

Case 1. $\frac{p-2}{3} \leq M \leq \frac{p-3}{2}$.

With $k = 2$ in Lemma 4.4, we have

$$p \leq M + 1 + |R_2|.$$

Applying Lemma 4.7 with $u = 0$ and $w = 6$, we have

$$|R_2| \leq p - 2M + 8.$$

It follows that $p \leq M + 1 + |R_2| = M + 1 + p - 2M + 8 < p$, a contradiction.

Case 2. $\frac{p+3}{4} \leq M \leq \frac{p-4}{3}$.

With $k = 3$ in Lemma 4.4, we have

$p \leq M + 1 + 2 + 2 + |R_2| + |R_3|$.

Applying Lemma 4.7 with $u = 1$ and $w = 6$, we have

$$|R_2| \leq \frac{p - 2M + 28}{3},$$
$$|R_3| \leq \frac{p - 3M + 20}{2}.$$

It follows that $p \leq M + 5 + \sum_{i=2}^{3} |R_i| = M + 5 + \frac{p-2M+28}{3} + \frac{p-3M+20}{2} < p$, a contradiction.

Case 3. $\frac{p-2}{5} \leq M \leq \frac{p+1}{4}$.

With $k = 4$ in Lemma 4.4, we have

$p \leq M + 1 + 2 \times 2 + 3 \times 2 + |R_2| + |R_3| + |R_4|$.

Applying Lemma 4.7 with $u = 1$ and $w = 6$, we have

$$|R_2| \leq \frac{p - 2M + 28}{3},$$
$$|R_3| \leq \frac{p - 3M + 20}{2},$$
$$|R_4| \leq \frac{p - 4M + 36}{3}.$$

It follows that $p \leq M + 11 + \frac{p-2M+28}{3} + \frac{p-3M+20}{2} + \frac{p-4M+36}{3} < p$, a contradiction.

Case 4. $\frac{p-1}{6} \leq M \leq \frac{p-3}{5}$.

With $k = 5$ in Lemma 4.4, we have

$p \leq M + 27 + \sum_{i=2}^{5} |R_i|$.

Applying Lemma 4.7 with $u = 1$ and $w = 6$, we have

$$|R_2| \leq \frac{p - 2M + 28}{3},$$

$$|R_3| \le \frac{p-3M+20}{2},$$
$$|R_4| \le \frac{p-4M+36}{3},$$
$$|R_5| \le \frac{p-5M+24}{2}.$$

Applying Lemma 4.6 with $t = 2, \ell = 3$ and $u = 12$, we have that either $|R_2| = 0$, or $|R_3| \le \frac{p-3M+55}{12}$. It follows that $|R_2| + |R_3| \le \max\{\frac{p-2M+28}{3} + \frac{p-3M+55}{12}, \frac{p-3M+20}{2}\} = \frac{5p-11M+167}{12}$.

It follows that $p \le M + 27 + \sum_{i=2}^{5} |R_i| = M + 27 + (|R_2| + |R_3|) + |R_4| + |R_5| \le \frac{5p-11M+167}{12} + \frac{p-4M+36}{3} + \frac{p-5M+24}{2} + 27 < p$, a contradiction.

Case 5. $\frac{p-5}{7} \le M \le \frac{p-5}{6}$.

With $k = 6$ in Lemma 4.4, we have

$p \le M + 37 + \sum_{i=2}^{6} |R_i|$.

Applying Lemma 4.7 with $u = 2$ and $w = 0$, we have

$$|R_2| \le \frac{p+18}{5},$$
$$|R_3| \le \frac{p-M+20}{4}.$$

Applying Lemma 4.7 with $u = 1$ and $w = 6$, we have

$$|R_4| \le \frac{p-4M+36}{3},$$
$$|R_5| \le \frac{p-5M+24}{2},$$
$$|R_6| \le \frac{p-6M+80}{5}.$$

It follows that $p \le M + 37 + \sum_{i=2}^{6} |R_i| = M + 37 + \frac{p+18}{5} + \frac{p-M+20}{4} + \frac{p-4M+36}{3} + \frac{p-5M+24}{2} + \frac{p-6M+80}{5} < p$, a contradiction.

Case 6. $\frac{p-2}{8} \le M \le \frac{p-3}{7}$.

With $k = 7$ in Lemma 4.4, we have

$p \le M + 73 + \sum_{i=2}^{7} |R_i|$.

Applying Lemma 4.7 with $u = 2$ and $w = 0$, we have

$$|R_2| \le \frac{p+18}{5},$$
$$|R_3| \le \frac{p-M+20}{4}.$$

Applying Lemma 4.7 with $u = 1$ and $w = 6$, we have

$$
\begin{aligned}
|R_4| &\le \frac{p-4M+36}{3},\\
|R_5| &\le \frac{p-5M+24}{2},\\
|R_6| &\le \frac{p-6M+80}{5},\\
|R_7| &\le \frac{p-7M+28}{2}.
\end{aligned}
$$

By Lemma 4.6, take $t=2$, $\ell=5$ and $u=10$, we have $|R_2|+|R_5| \le \max\{\frac{p-5M+4}{2}, \frac{p+18}{5} + \frac{p-5M-9}{10}+9\} = \frac{3p-5M+117}{10}$.

It follows that $p \le M+73+\sum_{i=2}^{7}|R_i| = M+73+(|R_2|+|R_5|)+|R_3|+|R_4|+|R_6|+|R_7| \le M+73+\frac{3p-5M+117}{10}+\frac{p-M+20}{4}+\frac{p-4M+36}{3}+\frac{p-6M+80}{5}+\frac{p-7M+28}{2} < p$, a contradiction.

Case 7. $\frac{p-2}{9} \le M \le \frac{p-3}{8}$.

With $k=8$ in Lemma 4.4, we have

$p \le M+111+\sum_{i=2}^{8}|R_i|$.

Applying Lemma 4.7 with $u=2$ and $w=0$, we have

$$
\begin{aligned}
|R_2| &\le \frac{p+18}{5},\\
|R_3| &\le \frac{p-M+20}{4},\\
|R_4| &\le \frac{p-2M+34}{5},\\
|R_5| &\le \frac{p-4M+22}{3}.
\end{aligned}
$$

Applying Lemma 4.7 with $u=1$ and $w=6$, we have

$$
\begin{aligned}
|R_6| &\le \frac{p-6M+80}{5},\\
|R_7| &\le \frac{p-7M+28}{2},\\
|R_8| &\le \frac{p-8M+52}{3}.
\end{aligned}
$$

Applying Lemma 4.6 with $t=2$, $\ell=5,7$ and $u=20$, we can prove that either $|R_2|=0$, or $|R_i| \le \frac{p-iM-2i+1}{20}+2i-1$ for both $i=5,7$. It follows that $|R_2|+|R_5|+|R_7| \le \max\{\frac{p-4M+22}{3}+\frac{p-7M+28}{2}, \frac{p-M+20}{4}+\frac{p-5M-9}{20}+9+\frac{p-7M-13}{20}+13\} = \frac{5p-29M+128}{6}$.

Applying Lemma 4.6 with $t=4, \ell=6$ and $u=10$, we obtain that either $|R_4|=0$, or $|R_6| \le \frac{p-6M-11}{10}+11$. It follows that $|R_4|+|R_6| \le \max\{\frac{p-2M+34}{5}+\frac{p-6M-11}{10}+11, \frac{p-6M+80}{5}\} = \frac{3p-10M+167}{10}$.

It follows that $p \le M+111+\sum_{i=2}^{8}|R_i| = M+111+(|R_2|+|R_5|+|R_7|)+(|R_4|+|R_6|)+|R_3|+|R_8| \le M+111+\frac{5p-29M+128}{6}+\frac{3p-10M+167}{10}+\frac{p-M+20}{4}+\frac{p-8M+52}{3} < p$, a contradiction.

Case 8. $\frac{p-2}{10} \le M \le \frac{p-4}{9}$.

With $k = 9$ in Lemma 4.4, we have

$p \le M + 159 + \sum_{i=2}^{9} |R_i|$.

Applying Lemma 4.7 with $u = 2$ and $w = 0$, we have

$$|R_2| \le \frac{p+18}{5},$$
$$|R_3| \le \frac{p-M+20}{4},$$
$$|R_4| \le \frac{p-2M+34}{5},$$
$$|R_5| \le \frac{p-4M+22}{3}.$$

Applying 4.7 with $u = 1$ and $w = 6$, we have

$$|R_6| \le \frac{p-6M+80}{5},$$
$$|R_7| \le \frac{p-7M+28}{2},$$
$$|R_8| \le \frac{p-8M+52}{3},$$
$$|R_9| \le \frac{p-9M+32}{2}.$$

Applying Lemma 4.6 with $t = 2, \ell = 5, 7$ and $u = 10$, we obtain that either $|R_2| = 0$, or $|R_i| \le \frac{p-iM-2i+1}{10} + 2i - 1$ for both $i = 5, 7$. It follows that $|R_2| + |R_5| + |R_7| \le \max\{\frac{p-4M+22}{3} + \frac{p-7M+28}{2}, \frac{p+18}{5} + \frac{p-5M-9}{10} + 9 + \frac{p-7M-13}{10} + 13\} = \frac{5p-29M+128}{6}$.

Applying Lemma 4.6 with $t = 3, \ell = 8$ and $u = 5$, we obtain that either $|R_3| = 0$, or $|R_8| \le \frac{p-8M-15}{8} + 15$. It follows that $|R_3| + |R_8| \le \max\{\frac{p-M+20}{4} + \frac{p-8M-15}{8} + 15, \frac{p-8M+52}{3}\} = \frac{3p-10M}{8} + 20$.

It follows that $p \le M + 159 + \sum_{i=2}^{9} |R_i| = M + 159 + (|R_2| + |R_5| + |R_7|) + (|R_3| + |R_8|) + |R_4| + |R_6| + |R_9| \le M + 159 + \frac{5p-29M+128}{6} + (\frac{3p-10M}{8} + 20) + \frac{p-2M+34}{5} + \frac{p-6M+80}{5} + \frac{p-9M+32}{2} < p$, a contradiction. □

5. The sum of four elements in Z_p

In this section, we shall give a new proof of the following result due to Li and Plyley [5].

Theorem 5.1. *Let p be a prime, and let S be a minimal zero-sum sequence of 4 elements in Z_p. Then $Index(S) = p$.*

For any real numbers $a \le b$, let $[a, b]$ denote the set of integers between a and b, $|[a, b]|$ denote the number of integers in $[a, b]$.

Define

$$S_{(p,j)} = \left\{ i : |ij|_p < \frac{p}{2}, i \in \left[1, \frac{p}{2}\right] \right\}$$

for every $j \in [1, p-1]$, where p is a prime.

It is easy to show

$$S_{(p,j)} \cap S_{(p,p-j)} = \emptyset \text{ and } S_{(p,j)} \cup S_{(p,p-j)} = \left[1, \tfrac{p}{2}\right]$$

for every $j \in [1, p-1]$.

Observation 5.2. $|S_{(p,j)}| + |S_{(p,p-j)}| = \frac{p-1}{2}$.

Lemma 5.3. *Let $p \geq 19$ be a prime and $j \in [2, p-2]$. Then $|S(p,j)| \geq \frac{p-1}{6}$ and we have strict inequality if $j \notin \{p-3, \frac{p-1}{3}\}$.*

Proof. Since $S_{(p,2)} = \left[1, \frac{p}{4}\right]$, we have $|S_{(p,2)}| = \left\lfloor \frac{p}{4} \right\rfloor \geq \frac{p-3}{4} > \frac{p-1}{6}$ and $|S_{(p,2)}| = \left\lfloor \frac{p}{4} \right\rfloor \leq \frac{p-1}{4} < \frac{p-1}{3}$, by Observation 5.2, $|S_{(p,p-2)}| > \frac{p-1}{2} - \frac{p-1}{3} = \frac{p-1}{6}$.

Since $S_{(p,3)} = \left[1, \frac{p}{6}\right] \cup \left[\frac{p}{3}, \frac{p}{2}\right]$, we have $|S_{(p,3)}| = \left\lfloor \frac{p}{6} \right\rfloor + \left\lfloor \frac{p}{2} \right\rfloor - \left\lceil \frac{p}{3} \right\rceil + 1$. It is easy to verify that $\frac{p-1}{6} < |S_{(p,3)}| \leq \frac{p-1}{3}$. By Observation 5.2, $|S_{(p,p-3)}| = \frac{p-1}{2} - |S_{(p,3)}| \geq \frac{p-1}{6}$.

It remains to consider $|S_{(p,j)}|$ and $|S_{(p,p-j)}|$ for $j \in [4, \frac{p}{2}]$.

Case 1. $4 \leq j = 2k < \frac{p}{2}$.

It is easy to show

$$S_{(p,2k)} = \left[1, \frac{p}{4k}\right] \cup \left[\frac{2p}{4k}, \frac{3p}{4k}\right] \cup \cdots \cup \left[\frac{(2k-2)p}{4k}, \frac{(2k-1)p}{4k}\right] \tag{12}$$

$$S_{(p,p-2k)} = \left[\frac{p}{4k}, \frac{2p}{4k}\right] \cup \left[\frac{3p}{4k}, \frac{4p}{4k}\right] \cup \cdots \cup \left[\frac{(2k-1)p}{4k}, \frac{(2k)p}{4k}\right]. \tag{13}$$

Subcase 1.1 $k \geq \frac{p-1}{6}$.

Noting that $\frac{p}{4k} > 1$, we have each interval contains at least one integer in (12) and (13). It follows that $|S_{(p,2k)}|, |S_{(p,p-2k)}| \geq k \geq \frac{p-1}{6}$, and moreover, equality implies $j = 2\frac{p-1}{6} = \frac{p-1}{3}$.

Subcase 1.2 $\frac{p-1}{6} > k > \frac{p}{8}$.

Then $\frac{p}{4k} < 2$. It follows that there are at most two integers in each interval in (12) and (13). It follows that $|S_{(p,2k)}|, |S_{(p,p-2k)}| \leq 2k < \frac{p-1}{3}$. By Observation 5.2, we have $|S_{(p,2k)}|, |S_{(p,p-2k)}| > \frac{p-1}{6}$.

Subcase 1.3 $\frac{p}{8} > k > \frac{p}{12}$.

Since $\frac{p}{4k} > 2$, we have that each interval in (12) and (13) contains at least two integers. Then $|S_{(p,2k)}|, |S_{(p,p-2k)}| \geq 2k > \frac{p-1}{6}$.

Subcase 1.4 $k < \frac{p}{12}$.

We see that each interval in (12) and (13) contains at most $\lfloor\frac{(i+1)p}{4k}\rfloor - \lceil\frac{ip}{4k}\rceil + 1 \le \frac{(i+1)p-1}{4k} - \frac{ip+1}{4k} + 1 = \frac{p-2}{4k} + 1$ integers, where $i = 0, 1, \ldots, 2k-1$. It follows that $|S_{(p,2k)}|, |S_{(p,p-2k)}| < k \times (\frac{p-2}{4k} + 1) < \frac{p-1}{3}$. By Observation 5.2, we have $|S_{(p,2k)}|, |S_{(p,p-2k)}| > \frac{p-1}{6}$.

Case 2. $5 \le j = 2k+1 < \frac{p}{2}$.

Observe that

$$S_{(p,2k+1)} = \left[1, \frac{p}{2(2k+1)}\right] \cup \left[\frac{2p}{2(2k+1)}, \frac{3p}{2(2k+1)}\right] \cup \cdots \cup \left[\frac{(2k)p}{2(2k+1)}, \frac{(2k+1)p}{2(2k+1)}\right] \tag{14}$$

$$S_{(p,p-2k-1)} = \left[\frac{p}{2(2k+1)}, \frac{2p}{2(2k+1)}\right] \cup \left[\frac{3p}{2(2k+1)}, \frac{4p}{2(2k+1)}\right] \cup \cdots \cup \left[\frac{(2k-1)p}{2(2k+1)}, \frac{(2k)p}{2(2k+1)}\right]. \tag{15}$$

Subcase 2.1 $k \ge \frac{p-1}{6}$.

Noting that $\frac{p}{2(2k+1)} > 1$, we have that at least one integer belongs to each interval in (14) and (15).

If $k > \frac{p-1}{6}$, then $|S_{(p,2k+1)}|, |S_{(p,p-2k-1)}| \ge k > \frac{p-1}{6}$. Hence, we may assume $k = \frac{p-1}{6}$.

Note that

$$\begin{aligned}\left[\frac{(2k-1)p}{2(2k+1)}, \frac{(2k)p}{2(2k+1)}\right] &= \left[\frac{(2k-1)(6k+1)}{2(2k+1)}, \frac{(2k)(6k+1)}{2(2k+1)}\right] \\ &= \left[3k - 2 - \frac{2k-3}{4k+2}, 3k - \frac{4k}{4k+2}\right],\end{aligned}$$

Since $k = \frac{p-1}{6} > 1$, we have that $3k-2, 3k-1 \in \left[\frac{(2k-1)p}{2(2k+1)}, \frac{(2k)p}{2(2k+1)}\right]$. It follows that $|S_{(p,2k+1)}|, |S_{(p,p-2k-1)}| \ge k+1 > \frac{p-1}{6}$.

Subcase 2.2 $\frac{p-1}{6} > k > \frac{p-4}{8}$.

Since p is a prime, we have $k \le \frac{p-5}{6}$.

Since $\frac{p}{2(2k+1)} < 2$, we have that at most two integers belong to each interval in (14) and (15). Moreover, there are just one integer in $\left[1, \frac{p}{2(2k+1)}\right]$. It follows that $|S_{(p,2k+1)}|, |S_{(p,p-2k-1)}| \le 2k+1 < \frac{p-1}{3}$.

Subcase 2.3 $\frac{p-4}{8} > k > \frac{p-1}{12}$.

Since $\frac{p}{2(2k+1)} > 2$, we have that at least two integers belong to each interval in (14) and (15). Then $|S_{(p,2k+1)}|, |S_{(p,p-2k-1)}| \ge 2k > \frac{p-1}{6}$.

Subcase 2.4 $\frac{p-1}{12} \ge k > \frac{p-1}{18}$.

If $k \in \{\frac{p-1}{12}, \frac{p-5}{12}\}$, then $2 < \frac{p}{2(2k+1)} < 3$, which implies that each interval in (14) and (15) contains at least two and at most three integers. It follows that $|S(p, 2k+1)| \ge (k+1) \times 2 > \frac{p-1}{6}$.

Moreover, since $\left[1, \frac{p}{2(2k+1)}\right]$ contains just two integers, it follows that $|S(p, 2k+1)| \le 2 + 3k < \frac{p-1}{3}$. By Observation 5.2, we have $|S(p, p-2k-1)| > \frac{p-1}{6}$.

Therefore, we assume $\frac{p-7}{12} \ge k > \frac{p-1}{18}$. Then $\frac{p}{2(2k+1)} > 3$. It follows that each interval in (14) and (15) contains at least three integers, which implies $|S_{(p,2k+1)}|, |S(p, p-2k-1)| \ge 3k > \frac{p-1}{6}$.

Subcase 2.5 $\frac{p-1}{18} \ge k > 1$.

Note that each interval in (14) and (15) contains $\left\lfloor \frac{(i+1)p}{2(2k+1)} \right\rfloor - \left\lceil \frac{ip}{2(2k+1)} \right\rceil + 1 \ge \frac{(i+1)p-2(2k+1)+1}{2(2k+1)} - \frac{ip+2(2k+1)-1}{2(2k+1)} + 1 = \frac{p-2(2k+1)+2}{2(2k+1)}$ integers, where $i = 1, 2, \ldots, 2k+1$. It follows that

$$|S_{(p,2k+1)}|, |S(p, p-2k-1)| \ge k \times \frac{p-2(2k+1)+2}{2(2k+1)}. \tag{16}$$

For $k = 2$, since $k \le \frac{p-1}{18}$, then $p \ge 37$. If $p > 43$, by (16), we have $|S_{(p,2k+1)}|, |S(p, p-2k-1)| > \frac{p-1}{6}$. If $p \in \{37, 41, 43\}$, it is easy to verify the conclusion.

Therefore, we may assume that $k \ge 3$. It follows from (16) that

$$|S_{(p,2k+1)}|, |S(p, p-2k-1)| > \frac{p-1}{6}.$$

This completes the proof. $\square$

Proof of Theorem 5.1. For $p < 19$, it is easy to check the theorem directly. So, we may assume $p \ge 19$. Let $S = a_1 \cdot a_2 \cdot a_3 \cdot a_4$, where $1 = a_1 \le a_2 \le a_3 \le a_4 < p$.

If $h(S) \ge 2$, say $a_1 = a_2 = 1$, then $a_1 + a_2 + a_3 + a_4 < 1 + 1 + (p-2) + (p-2) < 2p$, which implies that $a_1 + a_2 + a_3 + a_4 = p$. Hence, we assume

$$h(S) = 1.$$

If $S_{(p,a_2)} \cap S_{(p,a_3)} \cap S_{(p,a_4)} \ne \emptyset$, i.e., there exists $r \in S_{(p,a_2)} \cap S_{(p,a_3)} \cap S_{(p,a_4)}$, then $\sigma(|rS|_p) = |ra_1|_p + |ra_2|_p + |ra_3|_p + |ra_4|_p < 4 \times \frac{p}{2} = 2p$, which implies $\sigma(|rS|_p) = p$. Hence, we assume

$$S_{(p,a_2)} \cap S_{(p,a_3)} \cap S_{(p,a_4)} = \emptyset. \tag{17}$$

Since $h(S) = 1$, we have that at least one element of $\{a_2, a_3, a_4\}$ doesn't belong to $\{p-3, \frac{p-1}{3}\}$. It follows from Lemma 5.3 that

$$|S_{(p,a_2)}| + |S_{(p,a_3)}| + |S_{(p,a_4)}| > 3 \times \frac{p-1}{6} = \frac{p-1}{2}. \tag{18}$$

Since $\frac{p-1}{2} \ge |S_{(p,a_2)} \cup S_{(p,a_3)} \cup S_{(p,a_4)}| = |S_{(p,a_2)}| + |S_{(p,a_3)}| + |S_{(p,a_4)}| - (|S_{(p,a_2)} \cap S_{(p,a_3)}| + |S_{(p,a_3)} \cap S_{(p,a_4)}| + |S_{(p,a_2)} \cap S_{(p,a_4)}|) + |S_{(p,a_2)} \cap S_{(p,a_3)} \cap S_{(p,a_4)}|$, it follows from (17) and (18) that $|S_{(p,a_2)} \cap S_{(p,a_3)}| + |S_{(p,a_3)} \cap S_{(p,a_4)}| + |S_{(p,a_2)} \cap S_{(p,a_4)}| > 0$, say $r \in S_{(p,a_2)} \cap S_{(p,a_3)}$. By (17), $r \notin S_{(p,a_4)}$.

For $i = 1, 2, 3$, since $|ra_i|_p < \frac{p}{2}$, it follows that $|2ra_i|_p = 2|ra_i|_p$. Since $|ra_4|_p > \frac{p}{2}$, it follows that $|2ra_4|_p = 2|ra_4|_p - p$. Then $\sigma(|2rS|_p) = |2ra_1|_p + |2ra_2|_p + |2ra_3|_p + |2ra_4|_p = 2|ra_1|_p + 2|ra_2|_p + 2|ra_3|_p + 2|ra_4|_p - p$ is odd. It follows that $\sigma(|2rS|_p) \in \{p, 3p\}$. If $\sigma(|2rS|_p) = p$, we are done. If $\sigma(|2rS|_p) = 3p$, then $\sigma(|(p-2r)S|_p) = (p - |2ra_1|_p) + (p - |2ra_2|_p) + (p - |2ra_3|_p) + (p - |2ra_4|_p) = p$. This completes the proof of the theorem. $\square$

6. Remarks and Open Problems

It seems plausible to make the following generalization of Conjecture 1.2 when $n = p$ is a prime.

Conjecture 6.1. *If S is a sequence of p elements in Z_p then S contains a subsequence with index p and with length not exceeding $h(S)$.*

Let $t(n)$ be the smallest integer t such that every sequence of t elements in Z_n contains a subsequence of index n. From Theorem 1.6 we know that $t(n) \geq n + \lfloor \frac{n}{4} \rfloor - 4$ for $n = 4k+2 \geq 22$.

Open Problem 1 *To determine $t(n)$ for all positive integers n.*

Let $T(n)$ be the smallest integer t such that every subset of t distinct elements in Z_n contains a subset of index n.

Open Problem 2 *To determine $T(n)$ for all positive integers n.*

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